

ON NIKISHIN SYSTEMS WITH DISCRETE COMPONENTS AND WEAK ASYMPTOTICS OF MULTIPLE ORTHOGONAL POLYNOMIALS

A. I. APTEKAREV, G. LÓPEZ LAGOMASINO, AND A. MARTÍNEZ-FINKELSHTEIN

Dedicated to our teachers and friends Andrei Alexandrovich Gonchar and Herbert Stahl

ABSTRACT. We consider multiple orthogonal polynomials with respect to Nikishin systems generated by two measures (σ_1, σ_2) with unbounded supports ($\text{supp } \sigma_1 \subseteq \mathbb{R}_+$, $\text{supp } \sigma_2 \subseteq \mathbb{R}_-$) and σ_2 is discrete. A Nikishin type equilibrium problem in the presence of an external field acting on \mathbb{R}_+ and a constraint on \mathbb{R}_- is stated and solved. The solution is used for deriving the contracted zero distribution of the associated multiple orthogonal polynomials.

Keywords and phrases. Hermite-Padé approximants, multiple orthogonal polynomials, discrete orthogonality, weak asymptotic, vector equilibrium problem, Nikishin systems.

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1. INTRODUCTION

In a celebrated paper published in 1980, E.M. Nikishin [37] introduced a general class of systems of measures, now called Nikishin systems. Let $\Delta_\alpha, \Delta_\beta$ be two non-intersecting bounded intervals of the real line \mathbb{R} , and $\sigma_\alpha \in \mathcal{M}(\Delta_\alpha)$, $\sigma_\beta \in \mathcal{M}(\Delta_\beta)$, where $\mathcal{M}(\Delta)$ denotes the set of all finite Borel measures on the interval Δ with constant sign. With σ_α and σ_β we construct a third measure $\langle \sigma_\alpha, \sigma_\beta \rangle$, which using the differential notation is given by

$$(1) \quad d\langle \sigma_\alpha, \sigma_\beta \rangle(x) := \widehat{\sigma}_\beta(x) d\sigma_\alpha(x), \quad \widehat{\sigma}_\beta(x) = \int (x-t)^{-1} d\sigma_\beta(t).$$

DEFINITION 1.1. Take a collection Δ_j , $j = 1, \dots, m$, of intervals such that

$$\Delta_j \cap \Delta_{j+1} = \emptyset, \quad j = 1, \dots, m-1,$$

and a system of measures $(\sigma_1, \dots, \sigma_m)$ with $\sigma_j \in \mathcal{M}(\Delta_j)$, $j = 1, \dots, m$; we assume additionally that for each j , the convex hull of the support $\text{supp}(\sigma_j)$ of σ_j coincides with Δ_j . Let

$$s_1 = \sigma_1, \quad s_1 = \langle \sigma_1, \sigma_2 \rangle, \quad \dots, \quad s_m = \langle \sigma_1, \langle \sigma_2, \dots, \sigma_m \rangle \rangle.$$

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We say that (s_1, \dots, s_m) is the *Nikishin system of measures* generated by $(\sigma_1, \dots, \sigma_m)$, and denote it by $(s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$.

This model system was introduced in order to study general properties of Hermite-Padé approximants and multiple orthogonal polynomials.

Fix $\mathbf{n} := (n_1, \dots, n_m) \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is the m dimensional zero vector. Define $P_{\mathbf{n}}$ as a non-zero polynomial of degree $\deg(P_{\mathbf{n}}) \leq |\mathbf{n}| := n_1 + \dots + n_m$ such that

$$\int x^\nu P_{\mathbf{n}}(x) ds_j(x) = 0, \dots, \quad \nu = 0, \dots, n_j - 1, \quad j = 1, \dots, m.$$

The existence of $P_{\mathbf{n}}$ reduces to solving a homogeneous linear system of $|\mathbf{n}|$ equations on the $|\mathbf{n}| + 1$ coefficients of $P_{\mathbf{n}}$; therefore, a non-trivial solution is guaranteed. However, in contrast with the scalar case ($m = 1$) of standard orthogonal polynomials (OP), uniqueness up to a constant factor is not a trivial matter (and, in general, not true for systems of arbitrary measures (s_1, \dots, s_m)). In connection with this question in [37] it was shown that in presence of a Nikishin system uniqueness holds, with $\deg P_{\mathbf{n}} = |\mathbf{n}|$, for multi-indices of the form $(n + 1, \dots, n + 1, n, \dots, n)$, and stated without proof that it is also true whenever $n_1 \geq \dots \geq n_m$. In the sequel we assume that $P_{\mathbf{n}}$ is monic.

Motivated by the structure of Nikishin systems, Herbert Stahl studied their analytic and algebraic properties (see [9]). In a series of papers [19]–[21], among other results, K. Driver and H. Stahl showed that uniqueness remains valid whenever $n_j \leq n_k + 1$, $1 \leq k < j \leq m$. The problem for arbitrary multi-indices was definitely solved in [25] (and [26] when the generating measures have unbounded and/or touching supports).

A remarkable property of Nikishin orthogonal polynomials is that they not only share orthogonality relations with respect to several measures but they also satisfy full orthogonality relations with respect to a single (varying with respect to \mathbf{n}) measure. For $m = 2$ and $n_2 \leq n_1 + 1$ this was first observed by Andrei Aleksandrovich Gonchar¹ by showing that the function of the second kind

$$R_{\mathbf{n},1}(z) = \int \frac{P_{\mathbf{n}}(x)}{z - x} d\sigma_1(x)$$

satisfies the orthogonality relations

$$(2) \quad \int x^\nu R_{\mathbf{n},1}(x) d\sigma_2(x) = 0, \quad \nu = 0, \dots, n_2 - 1.$$

From here it follows that $R_{\mathbf{n},1}$ has exactly n_2 zeros in $\mathbb{C} \setminus \Delta_1$, they are all simple, and lie in the interior of Δ_2 . If $P_{\mathbf{n},2}$ denotes the monic polynomial of degree n_2 vanishing at these points, then

$$(3) \quad \int x^\nu P_{\mathbf{n}}(x) \frac{d\sigma_1(x)}{P_{\mathbf{n},2}(x)} = 0, \quad \nu = 0, \dots, n_1 + n_2 - 1.$$

The study of the asymptotic behavior of multiple orthogonal polynomials is greatly indebted to A. A. Gonchar. In joint papers with E. A. Rakhmanov [27]–[29], they introduced the notion of vector equilibrium problem to describe the asymptotic zero distribution of such polynomials. For a Nikishin system of two measures and $n_1 = n_2 = n$ the result may be

¹On one of the regular Monday seminars at the Steklov Institute A. A. Gonchar was reporting on the results contained in [37] but after a short while he had to leave because of an important meeting he had to attend. After about an hour he returned and started anew his presentation proving (2) and (3) and from there deduced the convergence of the corresponding Hermite-Padé approximants.

stated as follows. Define the normalized zero counting measure ν_P of a polynomial P as

$$\nu_P = \frac{1}{\deg P} \sum_{P(x)=0} \delta_x,$$

where δ_x denotes the Dirac measure with mass 1 at the point x , and each zero of P is taken with account of its multiplicity, so that the total variation $|\nu_P|$ of ν_P is 1. Assume that $\sigma_j \in \mathbf{Reg}$, $j = 1, 2$ (for the definition of the class \mathbf{Reg} of measures, see [50, Chapter 3]). Then there exist positive measures $\lambda_j \in \mathcal{M}(\Delta_j)$, $j = 1, 2$, $|\lambda_1| = 2$, $|\lambda_2| = 1$, such that

$$(4) \quad \lim_n \nu_{P_n} = \lambda_1/2, \quad \lim_n \nu_{P_{n,2}} = \lambda_2,$$

in the weak-* topology of measures, where λ_1 and λ_2 are uniquely determined by the solution of the vector equilibrium problem

$$(5) \quad \begin{aligned} 2\mathcal{P}^{\lambda_1}(x) - \mathcal{P}^{\lambda_2}(x) &\begin{cases} = \gamma_1, & x \in \text{supp}(\lambda_1), \\ \geq \gamma_1, & x \in \Delta_1 \setminus \text{supp}(\lambda_1), \end{cases} \\ \mathcal{P}^{\lambda_2}(x) - \mathcal{P}^{\lambda_1}(x) &\begin{cases} = \gamma_2, & x \in \text{supp}(\lambda_2), \\ \geq \gamma_2, & x \in \Delta_2 \setminus \text{supp}(\lambda_2), \end{cases} \end{aligned}$$

and \mathcal{P}^λ denotes the logarithmic potential of λ (see the definition below). At the time, this result and its extensions were well known within a small circle of specialists. With some variations, for general Nikishin systems it appeared in papers by H. Stahl [49], and with the highest degree of generality by A. A. Gonchar, E. A. Rakhmanov, and V. N. Sorokin [30]. For other extensions and generalizations see [5], [7], [12], [15], [24], [38], [42], [43].

In recent years, Nikishin systems have attracted new attention because this construction has been identified in different models of random matrix theory and multiple orthogonal polynomial ensembles, see [6], [32], and [33]. In some of these models new ingredients appear in which some of the generating measures turn out to be discrete and/or have unbounded support. V. N. Sorokin has studied the asymptotic distribution of the zeros for several multiple orthogonal polynomials of this type, see [46]–[47].

Orthogonal polynomials with respect to discrete measures have the characteristic that between two consecutive mass points there may be at most one zero of the polynomial. This fact induces a constraint on the equilibrium problem whose solution describes the asymptotic zero distribution of the orthogonal polynomials. This effect was first pointed out by E. A. Rakhmanov in [41] (see also [18] and [36]). A similar situation occurs in the case of multiple orthogonal polynomials.

The present paper is devoted to the study of multiple orthogonal polynomials with respect to Nikishin systems generated by two measures (σ_1, σ_2) with unbounded supports ($\text{supp}(\sigma_1) \subseteq \mathbb{R}_+$, $\text{supp}(\sigma_2) \subseteq \mathbb{R}_-$); moreover, the second measure σ_2 is discrete. To obtain the limiting zero distribution (4) of such multiple OP we state and solve a Nikishin type equilibrium problem which generalizes (5) by having an external field acting on \mathbb{R}_+ and a constraint on \mathbb{R}_- . The main results are stated in Section 2. In Section 3 we review some examples of explicit solutions of the type of equilibrium problems that we consider. The last two sections include the proofs of the main results.

2. STATEMENT OF THE MAIN RESULTS

Let $d\sigma_1(x) = \sigma_1'(x)dx$ be a positive and absolutely continuous measure on \mathbb{R}_+ , and σ_2 a purely discrete measure on \mathbb{R}_- given by

$$(6) \quad \sigma_2 = \sum_{k \geq 1} \beta_k \delta_{t_k}, \quad 0 > t_1 > t_2 > \cdots, \quad t_k \rightarrow -\infty, \quad \lambda_k > 0.$$

All the moments of σ_1 are assumed to be finite and $\widehat{\sigma}_2$ is integrable with respect to σ_1 . Let $(s_1, s_2) = \mathcal{N}(\sigma_1, \sigma_2)$ be the Nikishin system generated by these measures. For $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}_+^2 \setminus \{0\}$ we define $P_{\mathbf{n}}$ as the monic polynomial of degree $|\mathbf{n}|$ which satisfies

$$(7) \quad \int x^\nu P_{\mathbf{n}}(x) ds_j(x) = 0, \quad \nu = 0, \dots, n_j - 1, \quad j = 1, 2.$$

The zeros of $P_{\mathbf{n}}$ are simple, lie in the interior of \mathbb{R}_+ , and $\deg P_{\mathbf{n}} = n_1 + n_2$. We will restrict our attention to sequences of multi-indices of the form $\mathbf{n} = (n, n)$. In order to simplify the notation we write P_n instead of $P_{\mathbf{n}}$. Thus, $\deg P_n = 2n$. Our goal is to describe the (rescaled) asymptotic zero distribution of the polynomials (P_n) , $n \in \mathbb{N}$, under appropriate assumptions on the generating measures σ_j , $j = 1, 2$.

Using the properties of Nikishin systems (see [26] and [30]) it is easy to deduce that there exists a monic polynomial $P_{n,2}$, $\deg P_{n,2} = n$, whose zeros are simple and contained in the interior of the convex hull of $\text{supp}(\sigma_2)$, such that

$$(8) \quad \int x^\nu \frac{P_n(x)}{P_{n,2}(x)} d\sigma_1(x) = 0, \quad \nu = 0, \dots, 2n - 1,$$

and

$$(9) \quad \int t^\nu \frac{P_{n,2}(t)}{P_n(t)} \int \frac{P_n^2(x)}{P_{n,2}(x)} \frac{d\sigma_1(x)}{x - t} d\sigma_2(t) = 0, \quad \nu = 0, \dots, n - 1.$$

In other words, P_n and $P_{n,2}$ satisfy full orthogonality relations with respect to varying measures.

Let $(d_n)_{n \in \mathbb{Z}_+}$, $d_n \geq 1$, be a sequence of numbers, and let

$$(10) \quad Q_n(x) = P_n(d_n x) / d_n^{2n}, \quad Q_{n,2}(t) = P_{n,2}(d_n t) / d_n^n.$$

Making the change of variables $x \rightarrow d_n x$, $t \rightarrow d_n t$ it follows that the monic polynomials Q_n , $Q_{n,2}$ verify the orthogonality relations

$$(11) \quad \int x^\nu \frac{Q_n(x)}{Q_{n,2}(x)} \sigma_1'(d_n x) dx = 0, \quad \nu = 0, \dots, 2n - 1,$$

and

$$(12) \quad \int t^\nu \frac{Q_{n,2}(t)}{Q_n(t)} \int \frac{Q_n^2(x)}{Q_{n,2}(x)} \frac{\sigma_1'(d_n x) dx}{x - t} d\sigma_{2,n}(t) = 0, \quad \nu = 0, \dots, n - 1,$$

where

$$(13) \quad d\sigma_{2,n}(t) = \sum_{k \geq 1} \beta_k \delta_{\xi_{k,n}}(t), \quad \xi_{k,n} = t_k / d_n.$$

We must impose some restrictions on the points $\xi_{k,n}$ and the numbers β_k, d_n :

- (i) There exists a positive continuous function ρ on \mathbb{R}_- such that for every compact set $K \subset \mathbb{R}_-$,

$$|\xi_{k+1,n} - \xi_{k,n}| > \rho(\xi_{k,n}) / n, \quad k \geq 1.$$

(ii) There exist two positive functions $A(x)$, $B(n)$, such that

$$\#\{k : \xi_{k,n} \in [x, 0]\} \leq A(x)B(n), \quad x < 0,$$

where A and B satisfy that for every $\beta > 0$,

$$\lim_{x \rightarrow -\infty} \frac{\log A(x)}{|x|^\beta} = 0 \quad \text{and} \quad \lim_n \frac{\log B(n)}{n} = 0.$$

(iii) For each fixed $x < 0$

$$\lim_n (\inf\{\beta_k : \xi_{k,n} \in [x, 0]\})^{1/n} = 1.$$

(iv) There exists a finite positive Borel measures σ supported on \mathbb{R}_- , $|\sigma| > 1$, such that for every compact subset $K \subset \mathbb{R}_-$, the logarithmic potential $\mathcal{P}^{\sigma|_K}$ (see the definition in (17)) of the restriction of σ to K is continuous on \mathbb{C} and

$$(14) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \int \sum_{k \geq 0} f(x) \delta_{\xi_{k,n}} = \int f(x) d\sigma(x)$$

for every bounded continuous function f with compact support in \mathbb{R}_- .

(v) There exists a continuous function φ on \mathbb{R}_+ such that

$$(15) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \sigma'_1(d_n x) = -\varphi(x)$$

uniformly on each compact subset of \mathbb{R}_+ , for which the following condition holds:

$$(16) \quad \lim_{x \rightarrow +\infty} (\varphi(x) - 4 \log x) = +\infty.$$

In order to describe the zero asymptotic behaviour of the multiple orthogonal polynomials Q_n , $Q_{n,2}$, we need to solve an associated vector equilibrium problem that we now present.

For a closed subset $\Delta \subset \mathbb{R}$ we denote by $\mathcal{M}^+(\Delta)$ the class of all finite positive Borel measures μ such that $\text{supp}(\mu) \subset \Delta$. We write $\mu \in \mathcal{M}_c^+(\Delta)$ if, additionally, $|\mu| = c$. Let $\mu \in \mathcal{M}^+(\mathbb{R})$. Its logarithmic potential and energy are defined as

$$(17) \quad \mathcal{P}^\mu(x) := \int \log \frac{1}{|x-y|} d\mu(y), \quad I(\mu) := \int \int \log \frac{1}{|x-y|} d\mu(x) d\mu(y),$$

respectively, whenever these integrals are well defined.

Assume that $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R})$ verify

$$(18) \quad I(\mu) < +\infty, \quad \int \log(1 + |x|^2) d\mu(x) < +\infty.$$

Their mutual energy may be defined as

$$I(\mu_1, \mu_2) := \int \int \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y).$$

Analogously, one can define the potential, energy, and mutual energy of signed measures. In particular, if (18) takes place then

$$I(\mu_1 - \mu_2) = I(\mu_1) + I(\mu_2) - 2I(\mu_1, \mu_2).$$

Moreover, for $\mu_1, \mu_2 \in \mathcal{M}_c^+(\mathbb{R})$, we have

$$(19) \quad I(\mu_1 - \mu_2) \geq 0,$$

with equality if and only if $\mu_1 = \mu_2$ (see [45], also [48] if the measures have a bounded support).

As in (iv) above, let $\sigma \in \mathcal{M}^+(\mathbb{R}_-)$, $\text{supp}(\sigma) = \mathbb{R}_-$, $|\sigma| > 1$, be such that for every compact subset $K \subset \mathbb{R}_-$ we have that $\mathcal{P}^{\sigma|_K}$ is continuous on \mathbb{C} (recall that $\sigma|_K$ denotes the restriction of σ to K). We define

$$(20) \quad \mathfrak{M}(\sigma) := \{\vec{\mu} = (\mu_1, \mu_2)^t \in \mathcal{M}_2^+(\mathbb{R}_+) \times \mathcal{M}_1^+(\mathbb{R}_-) : \mu_2 \leq \sigma\},$$

where the superscript t stands for transpose. By $\mu_2 \leq \sigma$ we mean that $\sigma - \mu_2$ is a positive measure. Since we have assumed that $\mathcal{P}^{\sigma|_K}$ is continuous on \mathbb{C} for every compact K , it readily follows that \mathcal{P}^{μ_2} is continuous on \mathbb{C} . Let φ be a continuous function on \mathbb{R}_+ satisfying (16). Unless otherwise stated, in what follows we assume that σ and φ verify all the properties just described.

Set

$$\mathfrak{M}^*(\sigma) := \{\vec{\mu} \in \mathfrak{M}(\sigma) : \int \log(1 + y^2) d\mu_1(y) < +\infty, \int \log(1 + y^2) d\mu_2(y) < +\infty\},$$

$$J_\varphi := \inf\{J_\varphi(\vec{\mu}) : \vec{\mu} \in \mathfrak{M}^*(\sigma)\}, \quad J_\varphi(\vec{\mu}) = 2 \left(I(\mu_1) - I(\mu_1, \mu_2) + I(\mu_2) + \int \varphi d\mu_1 \right),$$

and

$$W_1^{\vec{\mu}}(x) := 2\mathcal{P}^{\mu_1}(x) - \mathcal{P}^{\mu_2}(x) + \varphi(x), \quad W_2^{\vec{\lambda}}(x) := 2\mathcal{P}^{\lambda_2}(x) - \mathcal{P}^{\lambda_1}(x).$$

THEOREM 2.1. *With the assumptions on φ and σ above, the following statements are equivalent and have the same unique solution:*

(A) *There exists $\vec{\lambda} \in \mathfrak{M}^*(\sigma)$ such that $J_\varphi(\vec{\lambda}) = J_\varphi$.*

(B) *There exists $\vec{\lambda} \in \mathfrak{M}^*(\sigma)$ such that for all $\vec{\nu} \in \mathfrak{M}(\sigma)$,*

$$\int W_1^{\vec{\lambda}} d(\nu_1 - \lambda_1) + \int W_2^{\vec{\lambda}} d(\nu_2 - \lambda_2) \geq 0.$$

(C) *There exist $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{M}^*(\sigma)$ and constants w_1, w_2 such that*

$$(21) \quad 2\mathcal{P}^{\lambda_1}(x) - \mathcal{P}^{\lambda_2}(x) + \varphi(x) \begin{cases} = w_1, & x \in \text{supp}(\lambda_1), \\ \geq w_1, & x \in \mathbb{R}_+, \end{cases}$$

$$(22) \quad 2\mathcal{P}^{\lambda_2}(x) - \mathcal{P}^{\lambda_1}(x) \begin{cases} \leq w_2, & x \in \text{supp}(\lambda_2), \\ \geq w_2, & x \in \text{supp}(\sigma - \lambda_2). \end{cases}$$

The constants w_1, w_2 are uniquely determined as well. We also have that $\mathcal{P}^{\lambda_1}, \mathcal{P}^{\lambda_2}$ are continuous on \mathbb{C} , $\text{supp}(\lambda_1)$ is compact, and $\text{supp}(\lambda_2)$ is connected. If $x\varphi'(x) > 0$ is increasing on \mathbb{R}_+ then $\text{supp}(\lambda_1)$ is also connected. Should $\text{supp}(\lambda_2) \cap \text{supp}(\sigma - \lambda_2)$ be unbounded, we have $w_2 = 0$.

Results of this nature (in a more general setting regarding the dimension of the vector equilibrium problem and the supports of the corresponding measures) may be seen in [11]. There, the action of constraints on the measures is not considered and the external fields, which verify restrictions of the form (16), act on all the components of the vector measures. This implies in turn that all the components of the equilibrium vector measure have compact support. However, taking into consideration certain applications, we are especially interested in allowing the second component of the equilibrium measure to be unbounded. For this reason, in the proof of Theorem 2.1 (see also Lemma 4.1) we follow the approach presented in [31] where results similar to Theorem 2.1, except for part (C), also appear. We wish to point out that Theorem 2.1 remains valid if $\text{supp}(\sigma)$ is any non-trivial closed interval contained in \mathbb{R}_- .

Now we are ready to formulate the main result about the zero asymptotics of Nikishin orthogonal polynomial.

THEOREM 2.2. *Let the assumptions (i)–(v) formulated above hold, and let $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathcal{M}^*(\sigma)$ be the solution of the extremal problem in Theorem 2.1. Then*

$$\lim_n \nu_{Q_n} = \lambda_1/2, \quad \lim_n \nu_{Q_{n,2}} = \lambda_2,$$

in the weak- topology of measures. Also*

$$(23) \quad \lim_n \left| \int \frac{Q_n^2(x)}{Q_{n,2}(x)} \sigma'_1(d_n x) dx \right|^{1/n} = e^{-w_1},$$

and

$$(24) \quad \lim_n \left| \int \frac{Q_{n,2}^2(t)}{Q_n(t)} \int \frac{Q_n^2(x)}{Q_{n,2}(x)} \frac{\sigma'_1(d_n x) dx}{x-t} d\sigma_{2,n}(t) \right|^{1/n} = e^{-(w_1+w_2)},$$

where w_1, w_2 are the corresponding equilibrium constants from (21)–(22).

Although the assumptions of this theorem may seem too restrictive, it encompasses many interesting examples. Some of them will be discussed in the next Section. In particular, we will analyze briefly the case of the modified Bessel weights (appearing in the analysis of the non-intersecting squared Bessel paths), the multiple Hermite polynomials (useful when studying ensembles of random matrices with an external source), and finally, the multiple Pollaczek polynomials, studied previously in [46], which will be discussed in more detail, and for which an alternative method for solving the equilibrium problem of Theorem 2.1 is presented.

Let us finish this section noting that we can easily translate the results of Theorems 2.1 and 2.2 to the equivalent setting of the whole real axis \mathbb{R} (with symmetric measures with respect to the origin). Indeed, let $\{P_m\}$ be a sequence of multiple orthogonal polynomials satisfying (7) with respect to a Nikishin system (8)–(9) on the semiaxis \mathbb{R}_+ , and define the polynomial sequence $\{\tilde{P}_n\}$ with polynomials of even degrees by

$$(25) \quad \tilde{P}_n(x) := P_m(x^2), \quad m = \frac{n}{2}, \quad n \in 2\mathbb{N}.$$

Then \tilde{P}_n are multiple orthogonal polynomials satisfying conditions of the form (7) with respect to what can be seen as a natural generalization of a Nikishin system: now the first generating measure σ_1 is supported on the whole real axis \mathbb{R} , while the second generating measure σ_2 is a discrete measure on the imaginary axis. Then for the rescaled polynomials $\tilde{Q}_n(x) := P_m(d_m x^2)/d_m^{2m}$ we have straightforward analogues of Theorems 2.1 and 2.2, but now in terms of the solution of the following equilibrium problem: there exists a unique pair of measures (λ_1, λ_2) , $|\lambda_1| = 2$, $|\lambda_2| = 1$, and unique constants γ_1, γ_2 , such that $\lambda_2(x) \leq \tilde{\sigma}$ for $x \in i\mathbb{R}$,

$$(26) \quad 2\mathcal{P}^{\lambda_1}(x) - \mathcal{P}^{\lambda_2}(x) + \tilde{\varphi} \begin{cases} = \gamma_1, & x \in \text{supp}(\lambda_1) \subset \mathbb{R}, \\ \geq \gamma_1, & x \in \mathbb{R}, \end{cases}$$

$$(27) \quad 2\mathcal{P}^{\lambda_2}(x) - \mathcal{P}^{\lambda_1}(x) \begin{cases} \leq \gamma_2, & x \in \text{supp}(\lambda_2) \subset i\mathbb{R}, \\ \geq \gamma_2, & x \in \text{supp}(\sigma - \lambda_2). \end{cases}$$

The external field and the constraint are related to their analogues in (21)–(22) by $\tilde{\varphi}(x) = \varphi(x^2)$, $\tilde{\sigma}(x) = \sigma(x^2)$.

3. EXAMPLES OF EXPLICIT SOLUTIONS OF THE EQUILIBRIUM PROBLEM

As we already mentioned in the introduction, in recent years various models from random matrix theory have been reformulated in terms of multiple orthogonal polynomials corresponding to Nikishin systems of type (7)–(9). In all of them, the generated weights are given by entire functions whose ratio is a meromorphic function, which can be considered as the Cauchy transform of a discrete measure σ_2 as in (6).

In this section we discuss three examples of this type of Nikishin systems for which explicit solutions of the associated equilibrium problems stated in Theorem 2.1 are available. One of them (see subsection 3.3 below) is analyzed in more detail, along with a new approach for expressing the density of the equilibrium measure as a jump of the logarithm of an algebraic function. In this representation the constrained (by the Lebesgue measure) part of the equilibrium measure is modeled as the jump of the logarithm of a negative function. In contrast to the standard approach, where either the underlying differential equations or the recurrence relations of the corresponding multiple orthogonal polynomials are used, we derive this representation directly from the equilibrium conditions.

3.1. Modified Bessel weights (and non-intersecting squared Bessel paths). In [22], [23] multiple orthogonal polynomials $\{P_n\}$ satisfying (7) for the system of the weights

$$\begin{aligned} s'_1(x) &= x^{\alpha/2} e^{-\frac{x}{2}} I_\alpha(\sqrt{x}), \\ s'_2(x) &= x^{(\alpha+1)/2} e^{-\frac{x}{2}} I_{\alpha+1}(\sqrt{x}), \end{aligned} \quad x \in \mathbb{R}_+,$$

where I_α is the modified Bessel function, $\alpha > -1$, were introduced and studied. This system has found applications in the description of ensembles of particles following non-intersecting squared Bessel paths [33], [34].

The polynomials $\{P_n\}$, rescaled as in (10), allow for an application of the general Theorem 2.2. Therefore, their weak asymptotics is described by means of the extremal problem solved in Theorem 2.1, with a particular choice of the external field φ and the upper constraint σ ; namely,

$$(28) \quad \varphi(x) = \frac{x}{2} - \sqrt{x}, \quad x > 0, \quad \frac{d\sigma}{dx} = \frac{\sqrt{|x|}}{\pi}, \quad x < 0.$$

An explicit solution of the equilibrium problem (21)–(22) and (28) is known (see [33], or [6], page 1188). The measures λ_j , $j = 1, 2$, are absolutely continuous with respect to the Lebesgue measure with densities that can be expressed in terms of solutions of the cubic equation (a.k.a. the spectral curve)

$$(29) \quad H^3 - 2H^2 + H - \frac{2}{z} = 0.$$

Equation (29) has three solutions, enumerated in such a way that

$$\begin{aligned} H_0(z) &= \frac{2}{z} + O(z^{-2}), \\ H_1(z) &= 1 - \frac{\sqrt{2}}{z^{1/2}} - \frac{1}{z} + O(z^{-3/2}), \\ H_2(z) &= 1 + \frac{\sqrt{2}}{z^{1/2}} - \frac{1}{z} + O(z^{-3/2}), \end{aligned}$$

as $z \rightarrow \infty$. Then, as it was shown in [33], λ_1 and λ_2 can be written as

$$(30) \quad \begin{aligned} \lambda'_1(x) &= \frac{1}{\pi} \operatorname{Im} H_{0,+}(x), \quad x > 0, \\ \lambda'_2(x) &= \frac{d\sigma}{dx} - \frac{1}{\pi} \operatorname{Im} H_{1,+}(x), \quad x < 0, \end{aligned}$$

where the $+$ subindices indicate the boundary values from the upper half plane.

3.2. Multiple Hermite polynomials (and random matrices with an external source).

Another set of multiple orthogonal polynomials was described in [4]. It turns out that it is more convenient to deal with the polynomials $\{\tilde{P}_n\}$, defined by (25), with respect to the system of the weights

$$s'_j(x) = e^{-n(\frac{1}{2}x^2 - a_jx)}, \quad x \in \mathbb{R}, \quad j = 1, \dots, p.$$

This system has found applications in the description of ensembles of non-intersecting Brownian bridges or random matrices with external source [3], [14]. There, for the case $p = 2$ and $a_1 = -a_2 = a$, it was proved that the zero counting measures of the scaled polynomials $\{\tilde{Q}_n\}$ (corresponding to $\{\tilde{P}_n\}$) have a weak limit λ which can be described by means of the spectral curve

$$(31) \quad H^3 - zH^2 + (2 - a^2)H + za^2 = 0.$$

This equation is due to Pastur [40]. If we enumerate the branches in (31) so that, as $z \rightarrow \infty$,

$$\begin{aligned} H_0(z) &= z - \frac{2}{z} + O(z^{-2}), \\ H_1(z) &= a + \frac{1}{z} + O(z^{-2}), \\ H_2(z) &= -a + \frac{1}{z} + O(z^{-2}), \end{aligned}$$

then λ is given by

$$(32) \quad \lambda'(x) = \frac{1}{\pi} \operatorname{Im} H_{0,+}(x), \quad x \in \mathbb{R}.$$

It was noticed in [13] that the measure λ in (32) coincides with the component λ_1 in the solution of the equilibrium problem (26)–(27) corresponding to the external field $\tilde{\varphi}$ and the constraint $\tilde{\sigma}$ as follows:

$$\tilde{\varphi}(x) = \frac{x^2}{2} - a|x|, \quad x \in \mathbb{R}, \quad d\tilde{\sigma}(z) = \frac{a}{\pi}|dz|, \quad z \in i\mathbb{R}.$$

Actually, [13] contains a more general result for the multiple orthogonal polynomials $\{\tilde{P}_n\}$ given by (25), corresponding to the system of weights

$$s'_j(x) = e^{-(V(x) - a_jx)}, \quad x \in \mathbb{R}, \quad j = 1, 2,$$

where $V(x) = \sum_{j=1}^d v_j x^{2d}$ is an even polynomial potential with $v_d > 0$; it was shown that the zero counting measures of the scaled polynomials $\{\tilde{Q}_n\}$ (corresponding to $\{\tilde{P}_n\}$) converge (in a weak-* sense) to the first component $\lambda = \lambda_1$ of the solution to the equilibrium problem (26)–(27), with the external field $\tilde{\varphi}$ and the constraint $\tilde{\sigma}$ given by

$$(33) \quad \tilde{\varphi}(x) = V(x) - a|x|, \quad x \in \mathbb{R}, \quad d\tilde{\sigma}(z) = \frac{a}{\pi}|dz|, \quad z \in i\mathbb{R}.$$

Moreover, it was also proved in [13] that the equilibrium problem (26)–(27) with input data (33) has always a unique solution (λ_1, λ_2) , $|\lambda_1| = 2$, $|\lambda_2| = 1$, and that the functions

$$\begin{aligned} H_0(z) &= V'(z) - \int \frac{d\lambda_1(s)}{z-s}, & z \in \mathbb{C} \setminus S(\lambda_1), \\ H_1(z) &= \pm a + \int \frac{d\lambda_1(s)}{z-s} - \int \frac{d\lambda_2(s)}{z-s}, & z \in \mathbb{C} \setminus (S(\lambda_1) \cup S(\sigma - \lambda_2)), \quad \pm \operatorname{Re} z > 0, \\ H_2(z) &= \mp a + \int \frac{d\lambda_2(s)}{z-s}, & z \in \mathbb{C} \setminus S(\sigma - \lambda_2), \quad \pm \operatorname{Re} z > 0, \end{aligned}$$

are the three solutions of the equation

$$(34) \quad H^3 + p_2(z)H^2 + p_1(z)H + p_0(z) = 0$$

with polynomial coefficients, whose degrees can be easily determined from the degree of the potential V . However, finding the coefficients of these polynomials explicitly in the most general situation is a very difficult problem. In [8] (see also [13]) this was done for a general even quartic potential,

$$V(x) = \frac{1}{4}x^4 - \frac{b}{2}x^2$$

in the cases when the Riemann surface of (34) is of genus either 0 or 1. For instance, when the genus is 1 we have from [8] that

$$H^3 - (z^3 + bz)H^2 + z^2H + a^2z^3 = 0,$$

where a and b belong to the triangular domain on the (a, b) -plane, bounded by the curves

$$\begin{aligned} a_m(b) &:= \frac{\sqrt{6b^3 - 27b - 6(b^2 - 3)^{3/2}}}{9} > 0, & b \in (-2, -\sqrt{3}), \\ a_M(b) &:= \frac{\sqrt{6b^3 - 27b + 6(b^2 - 3)^{3/2}}}{9} > 0, & b \in (-\infty, -\sqrt{3}). \end{aligned}$$

and by the b -axis ($a = 0$).

3.3. Multiple Pollaczek polynomials. We have come to the main example of the application of the general Theorems 2.2 and 2.1 and of their analogues for the real axis \mathbb{R} , as discussed at the end of Section 2.

The sequence of polynomials, studied in [46], is defined by the multiple orthogonality conditions (7) on \mathbb{R}_+ with

$$(35) \quad ds_1(x) = \frac{dx}{\sinh \frac{\pi\sqrt{x}}{2}}, \quad ds_2(x) = \frac{1}{\cosh \frac{\pi\sqrt{x}}{2}} \frac{dx}{\sqrt{x}} = \frac{\tanh \frac{\pi\sqrt{x}}{2}}{\sqrt{x}} ds_1(x).$$

Decomposing $\tanh(\pi z/2)/z$ into simple fractions, it is easy to check that

$$\frac{\tanh \frac{\pi\sqrt{z}}{2}}{\sqrt{z}} = \frac{4}{\pi} \sum_{k \geq 0} \frac{1}{z + (2k+1)^2} = \int \frac{d\sigma_2(x)}{z-x}$$

where

$$\sigma_2 = \frac{4}{\pi} \sum_{k \in \mathbb{Z}_+} \delta_{-(2k+1)^2}$$

(cf. (6)). Hence, $(s_1, s_2) = \mathcal{N}(\sigma_1, \sigma_2)$ is a Nikishin system generated by $\sigma_1 = s_1$, supported on \mathbb{R}_+ , and the discrete measure σ_2 made of equal masses of size $4/\pi$, whose support is contained in $(-\infty, 0)$. In this case, the re-scaling (10) is done taking $d_n = 4n^2$. This yields the measure

$\sigma_{2,n}$, see (13), with $\xi_{k,n} = -((2k+1)/2n)^2$ and $\beta_k = 4/\pi$. It is easy to check that conditions (i)–(v) of Section 2 are satisfied with

$$(36) \quad \rho(x) = A(x) = \sqrt{|x|}, \quad d\sigma(x) = dx/\sqrt{|x|}, \quad \varphi(x) = \pi\sqrt{x}, \quad B(n) = n,$$

so that Theorem 2.2 can be applied.

Obviously, a pair of measures $(f ds_1, f ds_2)$, where f is any bounded positive continuous function on \mathbb{R}_+ , has associated the same vector equilibrium problem. Thus, the corresponding multiple orthogonal polynomials exhibit the same rescaled normalized zero distribution as those corresponding to (35). Other examples may be constructed replacing the discrete component of the Nikishin system by a Meixner or a Charlier type measure (see, for example, [36], [47] or [2]).

We will also consider the corresponding polynomials transplanted to the whole real axis, for multi-indices of the form (n, n) . Using the transformation (25) we obtain a sequence of monic polynomials \tilde{P}_n of degree $2n$, satisfying the orthogonality relations

$$(37) \quad \int_{\mathbb{R}} x^\nu \tilde{P}_n(x) \frac{xdx}{\sinh \pi x} = 0, \quad \nu = 0, \dots, n-1,$$

$$(38) \quad \int_{\mathbb{R}} x^\nu \tilde{P}_n(x) \frac{dx}{\cosh \pi x} = 0, \quad \nu = 0, \dots, n-1,$$

that are known as multiple (or generalized) Pollaczek polynomials (see [46]). In order to guarantee normality, we will assume additionally that the n are even. In this case, the zeros of \tilde{P}_n are real and simple.

In a similar fashion as it is done for Nikishin systems (on the real line) it can be deduced that there exists a monic polynomial $\tilde{P}_{n,2}$, $\deg \tilde{P}_{n,2} = n$, whose zeros are also simple and contained in $i\mathbb{R} \setminus \{0\}$, such that

$$(39) \quad \int_{\mathbb{R}} x^\nu \frac{\tilde{P}_n(x)}{\tilde{P}_{n,2}(x)} \frac{xdx}{\sinh(\pi x)} = 0, \quad \nu = 0, \dots, 2n-1,$$

and

$$(40) \quad \int_{\mathbb{R}} t^\nu \frac{\tilde{P}_{n,2}(t)}{\tilde{P}_n(t)} \int_{i\mathbb{R}} \frac{\tilde{P}_n^2(x)}{\tilde{P}_{n,2}(x)} \frac{xdx}{(x-t)\sinh(\pi x)} d\beta(t) = 0, \quad \nu = 0, \dots, n-1.$$

Set

$$\tilde{Q}_n(z) = \tilde{P}_n(nz)/n^{2n}, \quad \tilde{Q}_{n,2}(z) = \tilde{P}_{n,2}(nz)/n^n.$$

The logarithmic (weak) asymptotic behavior of these polynomials was studied by V. N. Sorokin in [46]. Sorokin's approach is based on the existence of an explicit expression of the generating function for the polynomials $\tilde{Q}_n(x)$, to which a weak form of the Darboux method can be applied. On this path, the weak asymptotics of the polynomials can be deduced from the singularities of the generating function.

By (36), the zero counting measures of the scaled polynomials $\{\tilde{Q}_n\}$ (corresponding to $\{\tilde{P}_n\}$) have a weak limit λ , which is the first component ($\lambda = \lambda_1$) of the solution to the equilibrium problem (26)–(27), with

$$(41) \quad \tilde{\varphi}(x) = \pi|x|, \quad x \in \mathbb{R}, \quad d\tilde{\sigma}(z) = |dz| \quad \text{on } i\mathbb{R}.$$

One of the goals of this section is to obtain λ by a direct solution of this equilibrium problem.

From electrostatic considerations we expect that $\text{supp}(\lambda_2) = i\mathbb{R}$, because the external field created by \mathcal{P}^{λ_1} on $i\mathbb{R}$ is too weak to make $\text{supp}(\lambda_2)$ compact. An alternative argument is that, if there were no restrictions on λ_2 , the measure $2\lambda_2$ in (27) would coincide with the balayage

of λ_1 onto $i\mathbb{R}$. Hence, the upper constraint forces the balayage measure to redistribute its mass precisely where it exceeds σ in order to attain equilibrium on the rest of $i\mathbb{R}$. This consideration makes us look for a solution λ_2 for which there is an equality on $\text{supp}(\sigma - \lambda_2)$ in the equilibrium conditions (27).

We shall try to find the Cauchy transform of the equilibrium measure λ_1 ,

$$(42) \quad H(z) := -\widehat{\lambda}_1(z) = \int_{\mathbb{R}} \frac{d\lambda_1(x)}{x - z}.$$

If we “complexify” the equilibrium relations (26)–(27) and (41), differentiate them and take the real parts, we obtain

$$\text{Re} \left(2\widehat{\lambda}_1(x) \right) - \widehat{\lambda}_2(x) = \begin{cases} -\pi, & \text{on } \mathbb{R}_- \cap \text{supp}(\lambda_1), \\ \pi, & \text{on } \mathbb{R}_+ \cap \text{supp}(\lambda_1), \end{cases}$$

and

$$\text{Re} \left(2\widehat{\lambda}_2(x) - \widehat{\lambda}_1(x) \right) = 0, \quad \text{on } \text{supp}(\sigma - \lambda_2).$$

Using the Riemann–Schwartz symmetry principle, from the first relation we deduce that the function H can be continued analytically from both sides of the cut along $\mathbb{R}_- \cap \text{supp}(\lambda_1)$. Thus, H can be lifted to a Riemann surface, where

$$(43) \quad H(z) = \pi + \widehat{\lambda}_1(z) - \widehat{\lambda}_2(z) := H_1(z)$$

is considered on the next sheet. Analogously, H can be continued analytically from both sides of the cut along $\mathbb{R}_+ \cap \text{supp}(\lambda_1)$, so that

$$(44) \quad H(z) = -\pi + \widehat{\lambda}_1(z) - \widehat{\lambda}_2(z) := H_2(z)$$

is defined on another sheet of the same surface. Let us assume that the complete Riemann surface $\mathcal{R} = \{\overline{\mathcal{R}^{(j)}}\}_{j=0}^2$, $\overline{\mathcal{R}^{(j)}} = \overline{\mathbb{C}}$, has three sheets. With appropriate cuts we will have three branches of $H = \{H_j\}_{j=0}^2$, where $H_0(z) = -\widehat{\lambda}_1(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \text{supp}(\lambda_1)$, and (42)–(44) give us that, as $z \rightarrow \infty$,

$$(45) \quad \begin{aligned} H_0(z) &= -\frac{2}{z} + \dots \\ H_1(z) &= \pi + \frac{1}{z} + \dots \\ H_2(z) &= -\pi + \frac{1}{z} + \dots \end{aligned}$$

We make an ansatz that the function H can be found in the form

$$(46) \quad H(\zeta) = \frac{2}{i} \log \psi(\zeta) \quad \text{on} \quad \mathcal{R} \setminus \{\zeta \in \mathcal{R} : \psi(\zeta) \in \mathbb{R}_-\},$$

where ψ is a meromorphic function on the compact three sheeted Riemann surface \mathcal{R} . At this moment, \mathcal{R} is still unknown (should it exist); however, representation (46) and relations (45) yield that

$$(47) \quad \psi(\zeta) = \begin{cases} 1 - \frac{i}{\zeta} + \dots, & \zeta \rightarrow \infty^{(0)}, \\ i - \frac{1}{2\zeta} + \dots, & \zeta \rightarrow \infty^{(1)}, \\ -i + \frac{1}{2\zeta} + \dots, & \zeta \rightarrow \infty^{(2)}, \end{cases}$$

where $q^{(j)}$ denotes the point on $\mathcal{R}^{(j)}$ whose canonical projection on the plane is $q \in \overline{\mathbb{C}}$. We try to take ψ as the simplest meromorphic function which maps \mathcal{R} conformally onto $\overline{\mathbb{C}}$. The

inverse of this function is a rational function $\zeta = r(\psi)$. From the main term in the asymptotic expansion (47) we have that

$$\zeta = \frac{A}{\psi - 1} + \frac{B}{\psi - i} + \frac{A}{\psi + i},$$

and the second term gives us that

$$A = -i, \quad B = \frac{-1}{2}, \quad C = \frac{1}{2}.$$

Thus,

$$(48) \quad \zeta = -i \frac{\psi(\psi + 1)}{(\psi^2 + 1)(\psi - 1)}$$

or, what is the same,

$$(49) \quad \psi^3 + \frac{i - \zeta}{\zeta} \psi^2 + \frac{i + \zeta}{\zeta} \psi - 1 = 0.$$

The discriminant of (49) is equal to

$$16\zeta^4 - 44\zeta^2 - 1.$$

Therefore, the algebraic function has four branch points $\pm e_1$ and $\pm e_2$, where

$$e_1 = \frac{1}{4}\sqrt{22 - 10\sqrt{5}}, \quad e_2 = \frac{i}{4}\sqrt{-22 + 10\sqrt{5}}.$$

Taking into account (47) we fix the following sheet structure of \mathcal{R} (see Figure 1)

$$(50) \quad \mathcal{R}^{(0)} := \overline{\mathbb{C}} \setminus [-e_1, e_1], \quad \mathcal{R}^{(1)} := \overline{\mathbb{C}} \setminus ([-e_1, 0] \cup [-e_2, e_2]), \quad \mathcal{R}^{(2)} := \overline{\mathbb{C}} \setminus ([0, e_1] \cup [-e_2, e_2]).$$

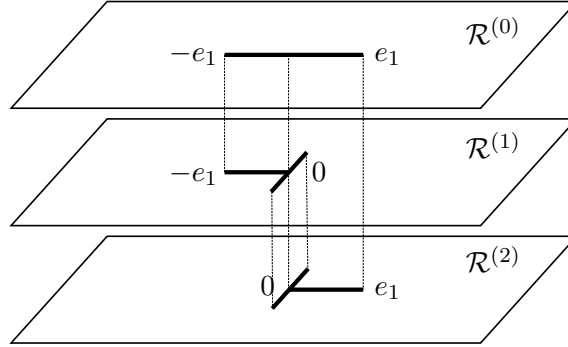


FIGURE 1. Sheet structure of the Riemann surface \mathcal{R} .

Therefore, the algebraic function ψ has the following single-valued meromorphic branches (in fact holomorphic, since $\psi(0) = \{0, -1, \infty\}$):

$$\begin{aligned} \psi_0(\zeta) &\in \mathcal{H}(\overline{\mathbb{C}} \setminus [-e_1, e_1]), & \psi_1(\zeta) &\in \mathcal{H}(\overline{\mathbb{C}} \setminus ([-e_1, 0] \cup [-e_2, e_2])), \\ \psi_2(\zeta) &\in \mathcal{H}(\overline{\mathbb{C}} \setminus ([0, e_1] \cup [-e_2, e_2])), \end{aligned}$$

where $\mathcal{H}(\Omega)$ stands for the class of functions holomorphic (and single-valued) in a domain Ω . From the analysis of the roots of (49) it follows that

$$(51) \quad \{i\mathbb{R}\}^{(0)} = \{\zeta \in \mathcal{R} : \psi(\zeta) \in \mathbb{R}_+\}, \quad \{[-e_2, e_2]\}^{(1)} \cup \{[-e_2, e_2]\}^{(2)} = \{\zeta \in \mathcal{R} : \psi(\zeta) \in \mathbb{R}_-\}.$$

Thus, if we cut our compact Riemann surface \mathcal{R} along the second set in (51) and denote

$$(52) \quad \tilde{\mathcal{R}} := \mathcal{R} \setminus (\{[-e_2, e_2]\}^{(1)} \cup \{[-e_2, e_2]\}^{(2)}),$$

we get that the function H in (46) is single-valued and holomorphic in the open Riemann surface $\tilde{\mathcal{R}}$. Now, we can formulate our result about the solution of the equilibrium problem (26)–(27):

PROPOSITION 3.1. *Let*

$$H_j(\zeta) = \frac{2}{i} \log \psi_j(\zeta), \quad \zeta \in \mathcal{R}^{(j)}, \quad j = 0, 1,$$

where the ψ_j are the solutions of (49) satisfying (47). Define the absolutely continuous measures

$$d\lambda_1(x) = \lambda'_1(x)dx, \quad d\lambda_2(x) = \lambda'_2(x)|dx|,$$

by

$$(53) \quad \begin{aligned} \lambda'_1(x) &= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0+} |\operatorname{Im} H_0(x + i\varepsilon)|, & x \in \mathbb{R}, \\ \lambda'_2(x) &= -1 + \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0+} \operatorname{Re} H_1(x - \varepsilon), & x \in i\mathbb{R} = \operatorname{supp}(\lambda_2). \end{aligned}$$

The pair (λ_1, λ_2) is the solution of the equilibrium problem (26)–(27) and (41). More precisely, $|\lambda_1| = 2$, $|\lambda_2| = 1$, and these measures verify

$$(54) \quad \text{for } d\sigma(x) = |dx|, \quad \lambda_2 \leq \sigma, \quad \text{and} \quad \lambda'_2(x) = 1 \text{ for } x \in [-e_2, e_2];$$

$$(55) \quad 2\mathcal{P}^{\lambda_1}(x) - \mathcal{P}^{\lambda_2}(x) + \pi|x| \begin{cases} = \gamma_1, & x \in [-e_1, e_1] = \operatorname{supp}(\lambda_1) \subset \mathbb{R}, \\ > \gamma_1, & x \in \mathbb{R}, \end{cases}$$

and

$$(56) \quad 2\mathcal{P}^{\lambda_2}(x) - \mathcal{P}^{\lambda_1}(x) \begin{cases} = \gamma_2, & x \in \operatorname{supp}(\sigma - \lambda_2) = i\mathbb{R} \setminus [-e_2, e_2], \\ < \gamma_2, & x \in [-e_2, e_2]. \end{cases}$$

Before proving Proposition 3.1 we discuss some properties of the primitive function G defined by

$$(57) \quad G' = H,$$

which we now consider on the open Riemann surface, $\tilde{\mathcal{R}}$. That is

$$(58) \quad G(\zeta) = \int_{\zeta_0}^{\zeta} H(t)dt, \quad \zeta_0, \zeta, t \in \tilde{\mathcal{R}}.$$

The uniformization of \mathcal{R} defined in (48) allows us to integrate by parts obtaining

$$(59) \quad G(\zeta) = -2 \int_{\psi(\zeta_0)}^{\psi(\zeta)} \log(\psi) d \frac{\psi(\psi+1)}{(\psi^2+1)(\psi-1)} = C + \zeta H(\zeta) + 2 \log(\psi(\zeta)-1) - \log(\psi^2(\zeta)+1),$$

where C is a constant which depends on ζ_0 . According to (59), G is multivalued on $\tilde{\mathcal{R}}$ and has local analytic extension to the whole \mathcal{R} (and beyond), with possible singular points at $\zeta = 0$ and $\zeta = \infty$ (notice that by (48), $\psi(\infty) = \{1, i, -i\}$). However, its periods are purely imaginary. Therefore, its real part is a single valued harmonic function on $\mathcal{R} \setminus \{0, \infty\}$,

$$g := \{g_j = \operatorname{Re} G_j\}_{j=0}^2,$$

which is defined up to an additive constant. We fix the constant so that

$$g_0(\infty) + g_1(\infty) + g_3(\infty) = 0.$$

This normalization in turn implies that

$$(60) \quad g_0(\zeta) + g_1(\zeta) + g_2(\zeta) \equiv 0, \quad \zeta \in \mathbb{C}.$$

Indeed, $g_0 + g_1 + g_2$ is a symmetric function of g which is harmonic on $\overline{\mathbb{C}} \setminus \{0, \infty\}$. From (48) and (59), one sees that the singularity it has at $\zeta = 0$ is removable. On the other hand, from (45) and (58), we have that the branches of g at infinity have the following behavior

$$(61) \quad g(\zeta) \simeq \begin{cases} -2 \log |\zeta|, & \zeta \rightarrow \infty^{(0)}, \\ \pi \operatorname{Re} z + \log |\zeta|, & \zeta \rightarrow \infty^{(1)}, \\ -\pi \operatorname{Re} z + \log |\zeta|, & \zeta \rightarrow \infty^{(2)}. \end{cases}$$

So, $\zeta = \infty$ is also a removable singularity of $g_0 + g_1 + g_2$. Since $g_0 + g_1 + g_2$ is harmonic in $\overline{\mathbb{C}}$ and equal to zero at ∞ , it is identically equal to zero.

Proof of Proposition 3.1. We must verify that the measures defined by their densities in (53) verify (54)–(56). In order to identify the potentials of the measures λ_1, λ_2 , let us change the sheet structure of \mathcal{R} . Define

$$(62) \quad g_0^* := g_0, \quad g_1^* := \begin{cases} g_1(z), & \operatorname{Re} z < 0, \\ g_2(z), & \operatorname{Re} z > 0, \end{cases} \quad g_2^* := \begin{cases} g_2(z), & \operatorname{Re} z < 0, \\ g_1(z), & \operatorname{Re} z > 0. \end{cases}$$

On $i\mathbb{R}$, g^* is defined by continuity. Notice that now g_1^*, g_2^* have a harmonic continuation through the interval $[-e_2, e_2]$.

Now, we see that the function g_0^* is superharmonic, and that g_2^* is subharmonic (being the maximum of two harmonic functions). Therefore, taking into account the behavior at ∞ (see (61)), from the Riesz decomposition theorem for superharmonic functions we obtain a global representation of the branches of g^* in \mathbb{C} in the form

$$(63) \quad \begin{aligned} g_0^*(z) &= \mathcal{P}^{\lambda_1}(z) + \kappa_1, \\ g_2^*(z) &= -\mathcal{P}^{\lambda_2}(z) - v(z) + \kappa_2, \end{aligned}$$

where λ_1, λ_2 are measures supported on $[-e_1, e_1]$ and $i\mathbb{R}$, respectively, and $v(z)$ is the superharmonic function

$$(64) \quad v(z) = \begin{cases} \pi \operatorname{Re} z, & \operatorname{Re} z \leq 0, \\ -\pi \operatorname{Re} z, & -\operatorname{Re} z > 0. \end{cases}$$

As a consequence of (60), we also have that

$$(65) \quad g_1^*(z) = -\mathcal{P}^{\lambda_1}(z) + \mathcal{P}^{\lambda_2}(z) + v(z) - \kappa_1 - \kappa_2.$$

Using (45) and (63), it is easy to verify that

$$|\lambda_1| = 2, \quad |\lambda_2| = 1,$$

and taking into consideration the definition of g , the Stieltjes-Perron formula applied to the calculation of the measures yields (53).

Since $g_0^*(x) = g_1^*(x)$ for $x \in [-e_1, e_1]$, using (63) and (65) we obtain the equality in (55) with $\gamma_1 := -2\kappa_1 - \kappa_2$. The fact that $g_0^*(x) > g_1^*(x)$ on $\mathbb{R} \setminus [-e_1, e_1]$ allows us to verify the inequality in (55). Analogously, comparing g_1^* and g_2^* on $i\mathbb{R}$, and using (63), (65) and the fact that $v(z) \equiv 0, z \in i\mathbb{R}$ (see (64)), we obtain (56) with $\gamma_2 := 2\kappa_1 + \kappa_2$.

Finally, notice that the functions ψ_1, ψ_2 have negative limiting values on $[-e_2, e_2]$ (see the second relation in (51)). Therefore, taking into consideration (46), it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{Re} H_1(x - \varepsilon) = 2\pi, \quad x \in [-e_2, e_2],$$

and $\lambda'_2(x) \equiv 1$, $x \in [-e_2, e_2]$. On the rest of the imaginary axis,

$$\pi < \lim_{\varepsilon \rightarrow 0+} \operatorname{Re} H_1(x - \varepsilon) < 2\pi$$

(see also (45)). Thus, we obtain (54). We wish to remark that when applying the Stieltjes-Perron formula in the second half of (53) we take the imaginary part because $|dx| = -idx$, $x \in i\mathbb{R}$. This concludes the proof. \square

4. PROOF OF THEOREM 2.1

In order to deal with the condition (18), not given a priori, we will employ the approach presented in [31]. For arbitrary $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R})$, we define a modified logarithmic potential and mutual energy as follows

$$\begin{aligned} \mathcal{V}^{\mu_1}(x) &:= \int \log \frac{\sqrt{1+y^2}}{|x-y|} d\mu_1(y), \\ \mathcal{I}(\mu_1, \mu_2) &:= \int \int \log \frac{\sqrt{1+x^2}\sqrt{1+y^2}}{|x-y|} d\mu_1(y) d\mu_2(x). \end{aligned}$$

The modified energy of μ is then given by $\mathcal{I}(\mu) := \mathcal{I}(\mu, \mu)$. The new kernel is connected with the inverse stereographic projection from the ball in \mathbb{R}^3 centered at $(0, 0, 1/2)$ and radius $1/2$ onto the extended complex plane. Therefore,

$$(66) \quad \frac{\sqrt{1+x^2}\sqrt{1+y^2}}{|x-y|} \geq 1$$

(for more details see (2.9)–(2.11) in [31]). Consequently, the modified potential and the mutual energy are uniformly bounded from below for all $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R})$. In case that μ_1, μ_2 fulfill (18),

$$\mathcal{I}(\mu_1, \mu_2) = I(\mu_1, \mu_2) + \frac{|\mu_2|}{2} \int \log(1+x^2) d\mu_1(x) + \frac{|\mu_1|}{2} \int \log(1+x^2) d\mu_2(x).$$

Let φ be a continuous function on \mathbb{R}_+ which verifies

$$(67) \quad \liminf_{x \rightarrow +\infty} (2\varphi(x) - 3\log(1+x^2)) > -\infty.$$

This assumption is much weaker than (16). Set $\varphi^*(x) := \varphi(x) - \frac{3}{2}\log(1+x^2)$, and define

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad f = \begin{pmatrix} \varphi^* \\ 0 \end{pmatrix}.$$

For $\vec{\mu} = (\mu_1, \mu_2)^t \in \mathfrak{M}(\sigma)$ (see the definition in (20)), we introduce the vector function

$$\mathcal{W}^{\vec{\mu}}(x) = (\mathcal{W}_1^{\vec{\mu}}(x), \mathcal{W}_2^{\vec{\mu}}(x))^t := \int \log \frac{\sqrt{1+x^2}\sqrt{1+y^2}}{|x-y|} dA\vec{\mu}(y) + f(x)$$

and the functional

$$(68) \quad \mathcal{J}_{\varphi^*}(\vec{\mu}) := \int (\mathcal{W}^{\vec{\mu}}(x) + f(x)) d\vec{\mu}(x) = \int (\mathcal{W}_1^{\vec{\mu}}(x) + \varphi^*(x)) d\mu_1(x) + \int \mathcal{W}_2^{\vec{\mu}}(x) d\mu_2(x)$$

(when either $\mathcal{I}(\mu_1) = +\infty$ or $\mathcal{I}(\mu_2) = +\infty$, we take $\mathcal{J}_{\varphi^*}(\vec{\mu}) = +\infty$). That is,

$$\mathcal{J}_{\varphi^*}(\vec{\mu}) = 2(\mathcal{I}(\mu_1) - \mathcal{I}(\mu_1, \mu_2) + \mathcal{I}(\mu_2)) + \int (2\varphi - 3\log(1+x^2)) d\mu_1.$$

Condition (67) guarantees that

$$\mathcal{J}_{\varphi^*} = \inf\{\mathcal{J}_{\varphi^*}(\vec{\mu}) : \vec{\mu} \in \mathfrak{M}(\sigma)\} > -\infty.$$

In case that $\vec{\mu} \in \mathfrak{M}^*(\sigma)$, it is easy to check that

$$(69) \quad \mathcal{J}_{\varphi^*}(\vec{\mu}) = 2(I(\mu_1) - I(\mu_1, \mu_2) + I(\mu_2) + \int \varphi d\mu_1) = J_{\varphi}(\vec{\mu}).$$

A vector measure $\vec{\lambda} \in \mathfrak{M}(\sigma)$ is said to be extremal if

$$\mathcal{J}_{\varphi^*}(\vec{\lambda}) = \mathcal{J}_{\varphi^*}.$$

The next lemma complements, in the present setting, results from [31].

LEMMA 4.1. *Let φ satisfy (67) and let $\sigma \in \mathcal{M}^+(\mathbb{R}_-)$, $\text{supp}(\sigma) = \mathbb{R}_-$, $|\sigma| > 1$, be such that for every compact subset $K \subset \mathbb{R}_-$ we have that $\mathcal{P}^{\sigma|_K}$ is continuous on \mathbb{C} . The following statements are equivalent and have the same unique solution:*

- (A) *There exists $\vec{\lambda} \in \mathfrak{M}(\sigma)$ such that $\mathcal{J}_{\varphi^*}(\vec{\lambda}) = \mathcal{J}_{\varphi^*}$.*
- (B) *There exists $\vec{\lambda} \in \mathfrak{M}(\sigma)$ such that for all $\vec{\nu} \in \mathfrak{M}(\sigma)$*

$$\int \mathcal{W}^{\vec{\lambda}} d(\vec{\nu} - \vec{\lambda}) := \int \mathcal{W}_1^{\vec{\lambda}} d(\nu_1 - \lambda_1) + \int \mathcal{W}_2^{\vec{\lambda}} d(\nu_2 - \lambda_2) \geq 0.$$

- (C) *There exist $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{M}(\sigma)$ and constants γ_1, γ_2 such that*

$$(i) \quad \mathcal{W}_1^{\vec{\lambda}}(x) = 2\mathcal{V}^{\lambda_1}(x) - \mathcal{V}^{\lambda_2}(x) + \varphi(x) \begin{cases} = \gamma_1, & x \in \text{supp}(\lambda_1), \\ \geq \gamma_1, & x \in \mathbb{R}_+, \end{cases}$$

$$(ii) \quad \mathcal{W}_2^{\vec{\lambda}}(x) = 2\mathcal{V}^{\lambda_2}(x) - \mathcal{V}^{\lambda_1}(x) \begin{cases} \leq \gamma_2, & x \in \text{supp}(\lambda_2), \\ \geq \gamma_2, & x \in \text{supp}(\sigma - \lambda_2). \end{cases}$$

The constants γ_1, γ_2 are uniquely determined as well. \mathcal{V}^{λ_1} and \mathcal{V}^{λ_2} are continuous on \mathbb{C} .

Proof. As shown in [31, Theorem 2.6], the functional \mathcal{J}_{φ^*} is lower semicontinuous and strictly convex on $\mathfrak{M}(\sigma)$, from which the existence of a unique solution to (A) is guaranteed by [31, Corollary 2.7]. Let us show that problems (A) and (B) are equivalent and have the same solution. Take $\vec{\lambda}, \vec{\nu} \in \mathfrak{M}(\sigma)$ and ε , $0 \leq \varepsilon \leq 1$. Define $\vec{\nu}_{\varepsilon} = \varepsilon\vec{\nu} + (1-\varepsilon)\vec{\lambda}$. Straightforward calculations yield

$$\mathcal{J}_{\varphi^*}(\vec{\nu}_{\varepsilon}) - \mathcal{J}_{\varphi^*}(\vec{\lambda}) = \varepsilon^2 \mathcal{J}_0(\vec{\nu} - \vec{\lambda}) + 2\varepsilon \int \mathcal{W}^{\vec{\lambda}} \cdot d(\vec{\nu} - \vec{\lambda}),$$

where $\mathcal{J}_0(\vec{\nu} - \vec{\lambda})$ is the functional applied to $\vec{\nu} - \vec{\lambda}$ with $\varphi^* \equiv 0$. Assume that $\vec{\lambda}$ is extremal. From the last formula it follows that

$$\varepsilon^2 \mathcal{J}_0(\vec{\nu} - \vec{\lambda}) + 2\varepsilon \int \mathcal{W}^{\vec{\lambda}} \cdot d(\vec{\nu} - \vec{\lambda}) \geq 0.$$

Dividing by ε and letting $\varepsilon \rightarrow 0$, we have

$$(70) \quad \int \mathcal{W}^{\vec{\lambda}} \cdot d(\vec{\nu} - \vec{\lambda}) \geq 0, \quad \vec{\nu} \in \mathfrak{M}(\sigma),$$

so (A) implies (B). Taking $\varepsilon = 1$, we get

$$\mathcal{J}_{\varphi^*}(\vec{\nu}) - \mathcal{J}_{\varphi^*}(\vec{\lambda}) = \mathcal{J}_0(\vec{\nu} - \vec{\lambda}) + 2 \int \mathcal{W}^{\vec{\lambda}} \cdot d(\vec{\nu} - \vec{\lambda}).$$

Since the matrix A is positive definite, it follows that $\mathcal{J}_0(\vec{\nu} - \vec{\lambda}) \geq 0$ with equality if and only if $\vec{\nu} = \vec{\lambda}$. For the proof see [31, Propositions 3.1, 3.5] (see also [16, Theorem 2.5]). Therefore, (B) implies (A).

If $\vec{\nu} \in \mathfrak{M}(\sigma)$ is also extremal, then

$$\mathcal{J}_0(\vec{\nu} - \vec{\lambda}) + 2 \int \mathcal{W}^{\vec{\lambda}} d(\vec{\nu} - \vec{\lambda}) = 0.$$

Taking (70) into consideration it follows that $\mathcal{J}_0(\vec{\nu} - \vec{\mu}) = 0$ and, consequently $\vec{\nu} = \vec{\lambda}$. Therefore, an extremal vector measure is unique.

Now, let us prove that any solution to (C) solves (B). Let $\vec{\lambda} = (\lambda_1, \lambda_2)^t$ verify (C) and take $\vec{\nu} = (\nu_1, \nu_2)^t \in \mathfrak{M}(\sigma)$. Notice that

$$\int \mathcal{W}^{\vec{\lambda}} d(\vec{\nu} - \vec{\lambda}) = \int \mathcal{W}_1^{\vec{\lambda}} d(\nu_1 - \lambda_1) + \int \mathcal{W}_2^{\vec{\lambda}} d(\nu_2 - \lambda_2).$$

From (C), condition (i), it follows that

$$\int \mathcal{W}_1^{\vec{\lambda}} d(\nu_1 - \lambda_1) = \int \mathcal{W}_1^{\vec{\lambda}} d\nu_1 - \int \mathcal{W}_1^{\vec{\lambda}} d\lambda_1 \geq \gamma_1 - \gamma_1 = 0.$$

On the other hand, $|\lambda_2| = |\nu_2| = 1$; therefore,

$$\int \mathcal{W}_2^{\vec{\lambda}} d(\nu_2 - \lambda_2) = \int (\mathcal{W}_2^{\vec{\lambda}} - \gamma_2) d(\nu_2 - \lambda_2).$$

Define

$$E_+ = \{t \in \text{supp}(\sigma) : \mathcal{W}_2^{\vec{\lambda}}(t) - \gamma_2 > 0\}, \quad E_- = \{t \in \text{supp}(\sigma) : \mathcal{W}_2^{\vec{\lambda}}(t) - \gamma_2 < 0\}.$$

According to (ii) in (C), $\lambda_2(E_+) = 0$, so

$$\int_{E_+} (\mathcal{W}_2^{\vec{\lambda}} - \gamma_2) d(\nu_2 - \lambda_2) \geq \int_{E_+} (\mathcal{W}_2^{\vec{\lambda}} - \gamma_2) d\nu_2 \geq 0.$$

Additionally, since $\nu_2 \leq \sigma$ and $(\sigma - \lambda_2)(E_-) = 0$,

$$\int_{E_-} (\mathcal{W}_2^{\vec{\lambda}} - \gamma_2) d(\nu_2 - \lambda_2) = \int_{E_-} (\mathcal{W}_2^{\vec{\lambda}} - \gamma_2) d(\nu_2 - \sigma) + \int_{E_-} (\mathcal{W}_2^{\vec{\lambda}} - \gamma_2) d(\sigma - \lambda_2) \geq 0.$$

Putting these relations together, we obtain

$$\int \mathcal{W}^{\vec{\lambda}} d(\vec{\nu} - \vec{\lambda}) \geq 0, \quad \nu \in \mathfrak{M}(\sigma),$$

as claimed. Therefore, (C) has a unique solution. It remains to show that (B) implies (C).

First, let us prove that \mathcal{V}^{λ_2} is continuous on \mathbb{C} . Obviously, \mathcal{V}^{λ_2} is continuous on $\mathbb{C} \setminus \mathbb{R}_-$. By Fatou's lemma it readily follows that it is lower semi-continuous on \mathbb{R}_- . Therefore, it is sufficient to show that it is also upper semi-continuous on this set. Choose $x_0 \leq 0$. If $x_0 < 0$, take $K = [-2x_0, 0]$; if $x_0 = 0$, take $K = [-1, 0]$. Then

$$\begin{aligned} \mathcal{V}^{\lambda_2}(x) &= \int_{\mathbb{R}_- \setminus K} \log \frac{\sqrt{1+y^2}}{|x-y|} d\lambda_2(y) \\ &\quad + \int_K \log \frac{\sqrt{1+y^2}}{|x-y|} d\sigma(y) - \int_K \log \frac{\sqrt{1+y^2}}{|x-y|} d(\sigma - \lambda_2)(y). \end{aligned}$$

The first term in the right hand side is obviously continuous at x_0 , which is at a positive distance from $\mathbb{R}_- \setminus K$; the second one is also continuous at x_0 , since by hypothesis $\mathcal{P}^{\sigma|_K}$ is continuous on \mathbb{R}_- . Finally, taking into consideration that $\sigma - \lambda_2$ is a positive measure, the last term (with the minus sign included) is upper semi-continuous on \mathbb{R}_- . Therefore, the sum of the three terms is upper semi-continuous at x_0 .

Set

$$\gamma_1 := \frac{1}{2} \int \mathcal{W}_1^{\vec{\lambda}} d\lambda_1.$$

Let us prove that

$$(71) \quad \mathcal{W}_1^{\vec{\lambda}}(x) \geq \gamma_1 \quad \text{quasi-everywhere on } \mathbb{R}_+,$$

where “quasi-everywhere” means except on a set of capacity zero. If this was not so, there would exist a compact subset $K_1 \subset \mathbb{R}_+$, $\text{cap}(K_1) > 0$, such that $\mathcal{W}_1^{\vec{\lambda}}(x) < \gamma_1$, $x \in K_1$. Taking $\nu_1 \in \mathcal{M}_2^+(\mathbb{R}_+)$, $\text{supp}(\nu_1) \subset K_1$, and $\nu_2 = \lambda_2$, we obtain

$$\int \mathcal{W}^{\vec{\lambda}} d(\vec{\nu} - \vec{\lambda}) = \int \mathcal{W}_1^{\vec{\lambda}} d(\nu_1 - \lambda_1) < 2\gamma_1 - 2\gamma_1 = 0,$$

which contradicts (B). Now, we prove that

$$\mathcal{W}_1^{\vec{\lambda}}(x) \leq \gamma_1, \quad x \in \text{supp}(\lambda_1).$$

To the contrary, assume that there exists $x_0 \in \text{supp}(\lambda_1)$ such that $\mathcal{W}_1^{\vec{\lambda}}(x_0) > \gamma_1$. By the lower semi-continuity of $\mathcal{W}_1^{\vec{\lambda}}$ on \mathbb{R}_+ (recall that \mathcal{V}^{λ_2} and φ are continuous) it follows that there exists $\delta > 0$ such that $\mathcal{W}_1^{\vec{\lambda}}(x) > \gamma_1$, $|x - x_0| \leq \delta$. Take $K_2 = \text{supp}(\lambda_1) \cap \{x : |x - x_0| \leq \delta\}$. Then $\lambda_1(K_2) > 0$ and

$$2\gamma_1 = \int_{\text{supp}(\lambda_1) \setminus K_2} \mathcal{W}_1^{\vec{\lambda}} d\lambda_1 + \int_{K_2} \mathcal{W}_1^{\vec{\lambda}} d\lambda_1 > \gamma_1(\lambda_1(\text{supp}(\lambda_1) \setminus K_2) + \lambda_1(K_2)) = 2\gamma_1,$$

which is also a contradiction. From (71), reasoning as in [39, Theorem 5.4.1], it follows that $\mathcal{W}_1^{\vec{\lambda}} \geq \gamma_1$ on all \mathbb{R}_+ . Hence, statement (i) of (C) is obtained. We have also obtained that $\mathcal{V}_1^{\lambda_1}$ is continuous on all \mathbb{C} since it is continuous on $\text{supp}(\lambda_1)$.

Set

$$\gamma_2 := \sup\{\gamma \in \mathbb{R} : \mathcal{W}_2^{\vec{\lambda}} \geq \gamma \quad (\sigma - \lambda_2) \text{ a.e.}\}.$$

Obviously, this supreme is a maximum. Suppose that there exists $x_0 \in \text{supp}(\lambda_2)$ such that $\mathcal{W}_2^{\vec{\lambda}}(x_0) > \gamma > \gamma_2$. By the definition of γ_2 , there exists a compact $K_1 \subset \text{supp}(\sigma - \lambda_2)$, such that $\mathcal{W}_2^{\vec{\lambda}}(x) < \gamma$, $x \in K_1$, and $(\sigma - \lambda_2)(K_1) > 0$. On the other hand, $\mathcal{W}_2^{\vec{\lambda}}(x)$ is lower semi-continuous on \mathbb{R}_- (in fact continuous), so there exists $\delta > 0$ sufficiently small such that $\mathcal{W}_2^{\vec{\lambda}}(x) > \gamma$ for $|x - x_0| < \delta$, and by the same token there exists a compact set K_2 with $\lambda_2(K_2) > 0$, such that $\mathcal{W}_2^{\vec{\lambda}}(x) > \gamma$ for $x \in K_2$. Obviously, $K_1 \cap K_2 = \emptyset$. Choose $\alpha, \beta \in (0, 1)$ such that $\beta(\sigma - \lambda_2)(K_1) = \alpha\lambda_2(K_2)$. Define a signed measure η equal to $-\alpha\lambda_2$ on K_2 , equal to $\beta(\sigma - \lambda_2)$ on K_1 , and zero otherwise. It is easy to check that $\lambda_2 + \eta$ is a positive measure of total mass 1 which is bounded by σ . Define $\vec{\nu} := (\lambda_1, \lambda_2 + \eta)^t \in \mathfrak{M}(\sigma)$. Then

$$\int \mathcal{W}^{\vec{\lambda}} d(\vec{\nu} - \vec{\lambda}) = \int \mathcal{W}_2^{\vec{\lambda}} d\eta < \gamma\beta(\sigma - \lambda)(K_1) - \gamma\alpha\lambda_2(K_2) = 0,$$

in contradiction with (B). From the continuity of $\mathcal{W}_2^{\vec{\lambda}}$ on \mathbb{C} , the inequality in the second part of (C) – (ii) holds for all $x \in \text{supp}(\sigma - \lambda_2)$. Therefore, (C) has been proved.

From the uniqueness of $\vec{\lambda}$ and the fact that $\text{supp}(\sigma - \lambda) \cap \text{supp}(\lambda_2) \neq \emptyset$ it readily follows that γ_1, γ_2 are uniquely determined. \square

LEMMA 4.2. *With the assumptions of Lemma 4.1, let $\vec{\lambda}$ be extremal. Then, $\text{supp}(\lambda_2)$ is connected. If $x\varphi'(x)$ is an increasing function on \mathbb{R}_+ then $\text{supp}(\lambda_1)$ is also connected. If*

$$(72) \quad \lim_{x \rightarrow +\infty} (\varphi(x) - 4 \log x) = +\infty,$$

then $\text{supp}(\lambda_1)$ is a compact set,

$$(73) \quad \int \log(1+y^2)d\lambda_1(y) < +\infty, \quad \int \log(1+y^2)d\lambda_2(y) < +\infty,$$

and in $(C-ii)$,

$$(74) \quad \gamma_2 = \int \log(1+y^2)d\lambda_2(y) - \frac{1}{2} \int \log(1+y^2)d\lambda_1(y).$$

Proof. Assume that $\text{supp}(\lambda_2)$ is not connected. Then, there exist $x_1, x_2 \in \text{supp}(\lambda_2)$, $x_2 < 0$, such that the interval $(x_1, x_2) \cap \text{supp}(\lambda_2) = \emptyset$. Straightforward calculations lead to

$$\frac{d}{dx} \left(x \left(\mathcal{W}_2^{\tilde{\lambda}}(x) \right)' \right) = 2 \int \frac{yd\lambda_2(y)}{(x-y)^2} + \int \frac{-yd\lambda_1(y)}{(x-y)^2} < 0, \quad x \in (x_1, x_2).$$

This implies that $(2\mathcal{V}^{\lambda_2}(x) - \mathcal{V}^{\lambda_1}(x))'$ cannot change sign from plus to minus on (x_1, x_2) , which contradicts $(C-ii)$.

On the other hand,

$$x \left(\mathcal{W}_1^{\tilde{\lambda}}(x) \right)' = 2x \int \frac{d\lambda_1(y)}{y-x} - x \int \frac{d\lambda_2(y)}{y-x} + x\varphi'(x)$$

and

$$\left(2x \int \frac{d\lambda_1(y)}{y-x} - x \int \frac{d\lambda_2(y)}{y-x} \right)' = 2 \int \frac{yd\lambda_1(y)}{(x-y)^2} + \int \frac{-yd\lambda_2(y)}{(x-y)^2} > 0, \quad x \in \mathbb{R}_+ \setminus \text{supp}(\lambda_1).$$

Therefore, if $x\varphi'(x)$ is increasing on \mathbb{R}_+ , the function $x \left(\mathcal{W}_1^{\tilde{\lambda}}(x) \right)'$ is increasing on any subinterval contained in $\mathbb{R}_+ \setminus \text{supp}(\lambda_1)$. Therefore, on any such subinterval $\left(\mathcal{W}_1^{\tilde{\lambda}}(x) \right)'$ cannot change sign from plus to minus and the connectedness of $\text{supp}(\lambda_1)$ readily follows using $(C-i)$.

Let us show that condition (72) implies that $\text{supp}(\lambda_1)$ is a compact set. Indeed, according to $(C-i)$,

$$\int \log \frac{\sqrt{1+y^2}}{|x-y|} d\frac{\lambda_1}{2}(y) - \int \log \frac{\sqrt{1+y^2}}{|x-y|} d\frac{\lambda_2}{4}(y) + \frac{\varphi(x)}{4} \begin{cases} = \gamma_1/4, & x \in \text{supp}(\lambda_1), \\ \geq \gamma_1/4, & x \in \mathbb{R}_+, \end{cases}$$

and (λ_1, λ_2) is the only pair of measures in $\mathfrak{M}(\sigma)$ satisfying (C) . The assumption on φ implies that

$$\lim_{x \rightarrow +\infty} \left(- \int \log \frac{\sqrt{1+y^2}}{|x-y|} d\frac{\lambda_2}{4}(y) + \frac{\varphi(x)}{4} - \log x \right) = +\infty.$$

According to [48, Theorem 1.1.3], there exists a unique probability measure $\tilde{\lambda}$ with compact support contained in \mathbb{R}_+ and a unique constant $\tilde{\gamma}$ such that

$$\mathcal{P}^{\tilde{\lambda}}(x) - \int \log \frac{\sqrt{1+y^2}}{|x-y|} d\frac{\lambda_2}{4}(y) + \frac{\varphi(x)}{4} \begin{cases} = \tilde{\gamma}, & x \in \text{supp}(\tilde{\lambda}), \\ \geq \tilde{\gamma}, & x \in \mathbb{R}_+. \end{cases}$$

From uniqueness, it follows that $\lambda_1 = 2\tilde{\lambda}$ has compact support and

$$\int \log(1+y^2)d\lambda_1(y) = \gamma_1 - 4\tilde{\gamma} < +\infty.$$

If $\text{supp}(\lambda_2)$ is compact, the other relation in (73) is immediate. Should $\text{supp}(\lambda_2)$ be unbounded, since $|\lambda_1| = 2, |\lambda_2| = 1$, from the first part of $(C - ii)$ we have

$$(75) \quad 2 \int \log \frac{\sqrt{1+y^2}}{|1-(y/x)|} d\lambda_2(y) - \int \log \frac{\sqrt{1+y^2}}{|1-(y/x)|} d\lambda_1(y) \leq \gamma_2, \quad x \in \text{supp}(\lambda_2).$$

This implies that

$$\limsup_{x \rightarrow \infty, x \in \text{supp}(\lambda_2)} 2 \int \log \frac{\sqrt{1+y^2}}{|1-(y/x)|} d\lambda_2(y) \leq \gamma_2 + \frac{1}{2} \int \log(1+y^2) d\lambda_1(y)$$

Using (66), we get

$$\frac{\sqrt{1+y^2}}{|1-(y/x)|} \geq \frac{|x|}{\sqrt{1+x^2}}.$$

Therefore, for all $x \leq -1$

$$\log \frac{\sqrt{1+y^2}}{|1-(y/x)|} \geq -(\log 2)/2.$$

Applying Fatou's lemma, we get

$$\begin{aligned} \int \log(1+y^2) d\lambda_2(y) &= 2 \int \liminf_{x \rightarrow \infty, x \in \text{supp}(\lambda_2)} \log \frac{\sqrt{1+y^2}}{|1-(y/x)|} d\lambda_2(y) \leq \\ \liminf_{x \rightarrow \infty, x \in \text{supp}(\lambda_2)} 2 \int \log \frac{\sqrt{1+y^2}}{|1-(y/x)|} d\lambda_2(y) &\leq \gamma_2 + \frac{1}{2} \int \log(1+y^2) d\lambda_1(y) < +\infty. \end{aligned}$$

With this we conclude the proof. \square

Proof of Theorem 2.1. The first part follows directly from Lemma 4.1 and (73). In fact, regarding statements (A) see (69). Then, (B) and (C) follow immediately using the connection between the potentials given by the formulas

$$2\mathcal{V}^{\lambda_1} - \mathcal{V}^{\lambda_2} + \varphi = 2\mathcal{P}^{\lambda_1} - \mathcal{P}^{\lambda_2} + \varphi + C_1, \quad 2\mathcal{V}^{\lambda_2} - \mathcal{V}^{\lambda_1} = 2\mathcal{P}^{\lambda_2} - \mathcal{P}^{\lambda_1} + C_2,$$

where

$$\begin{aligned} C_1 &= \int \log(1+y^2) d\lambda_1(y) - \frac{1}{2} \int \log(1+y^2) d\lambda_2(y), \\ C_2 &= \int \log(1+y^2) d\lambda_2(y) - \frac{1}{2} \int \log(1+y^2) d\lambda_1(y). \end{aligned}$$

Thus

$$w_1 = \gamma_1 - C_1, \quad w_2 = \gamma_2 - C_2.$$

The complementary statements of the theorem are direct consequence of the remaining assertions of Lemma 4.2.

Regarding the value of w_2 for the case when $\text{supp}(\lambda_2) \cap \text{supp}(\sigma - \lambda_2)$ is unbounded notice that in this case there is at least a sequence $(x_n)_{n \in \mathbb{Z}_+} \subset \text{supp}(\lambda_2) \cap \text{supp}(\sigma - \lambda_2)$, $\lim_n x_n = -\infty$ for which equality holds in (75). This implies, taking \liminf along this sequence of points and using Vitali's theorem (see [44, pp. 134-135]), that

$$\gamma_2 = \int \log(1+y^2) d\lambda_2(y) - \frac{1}{2} \int \log(1+y^2) d\lambda_1(y).$$

We are done. \square

5. PROOF OF THEOREM 2.2

Proof. Suppose that for some $\Lambda \subset \mathbb{N}$,

$$(76) \quad \lim_{n \in \Lambda} \nu_{Q_n} = \lambda_1^*, \quad \lim_{n \in \Lambda} \nu_{Q_{n,2}} = \lambda_2^*.$$

in the weak-* topology. That is

$$(77) \quad \lim_{n \in \Lambda} \int f d\nu_{Q_n} = \int f d\lambda_1^*, \quad \lim_{n \in \Lambda} \int f d\nu_{Q_{n,2}} = \int f d\lambda_2^*$$

for every bounded, continuous, real valued function f on \mathbb{R} . Since the zeros of Q_n and $Q_{n,2}$ lie in \mathbb{R}_+ and \mathbb{R}_- , respectively, it follows that $|\lambda_1^*| = 1$ in $\mathbb{R}_+ \cup \{+\infty\}$ and $|\lambda_2^*| = 1$ in $\mathbb{R}_- \cup \{-\infty\}$. Once we prove that λ_1^* and λ_2^* verify the second part of (18) we obtain that the limiting measures have no mass at ∞ and if $\lambda_2^* \leq \sigma$ we obtain that $(2\lambda_1^*, \lambda_2^*) \in \mathfrak{M}^*(\sigma)$. We also prove that $(2\lambda_1^*, \lambda_2^*) \in \mathfrak{M}^*(\sigma)$ solves the extremal problem (C) in Theorem 2.1. Then from unicity it follows that all convergent subsequences have the same limit. This basically settles Theorem 2.2.

Between two consecutive mass points of the discrete measure $\sigma_{2,n}$ there may be at most one zero of $Q_{n,2}$. Choose $T \in (-\infty, 0)$, then from (14) it follows that

$$\limsup_n \int_{[T,0]} d\nu_{Q_{n,2}} \leq \lim_n \frac{1}{n} \int_{[T,0]} \sum_{k \geq 0} d\delta_{\xi_{k,n}} = \int_{[T,0]} d\sigma.$$

This, together with the properties of σ , and the second part of (77) shows that $\lambda_2^*|_{\mathbb{R}_-} \leq \sigma$.

According to (11),

$$\int \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} C_n \sigma'(d_n x) dx \leq \int \frac{|Q(x)|^2}{|Q_{n,2}(x)|} C_n \sigma'(d_n x) dx, \quad C_n = \prod_{Q_{n,2}(x_k)=0} \sqrt{1+x_k^2},$$

for any monic polynomials Q , $\deg Q = 2n$. Notice that

$$\frac{1}{n} \log \frac{|Q_{n,2}(x)|}{C_n} = - \int \log \frac{\sqrt{1+y^2}}{|x-y|} d\nu_{Q_{n,2}}(y).$$

Since the kernel is bounded and continuous on \mathbb{R}_- for all $x > 0$, from (76) we obtain that

$$(78) \quad \lim_{n \in \Lambda} \frac{1}{2n} \log \left(\frac{|Q_{n,2}(x)|}{C_n} \right)^{1/2} = -\frac{1}{4} \int \log \frac{\sqrt{1+y^2}}{|x-y|} d\lambda_2^*(y) = -\frac{1}{4} \mathcal{V}^{\lambda_2^*}(x)$$

uniformly with respect to x on each compact subset of $\mathbb{R}_+ \setminus \{0\}$. For $x = 0$ we also have pointwise convergence since the kernel is uniformly bounded from below and we can use truncated kernels and Lebesgue's monotone convergence theorem to obtain convergence at that point. In fact, there is uniform convergence on any compact subset of \mathbb{R}_+ because $\lambda_2^*|_{\mathbb{R}_-} \leq \sigma$ and we can prove that $\mathcal{V}^{\lambda_2^*}$ is continuous on \mathbb{C} as we did to prove the same property for \mathcal{V}^{λ_2} . (Notice that the kernel takes value zero at $-\infty$ for any $x \in \mathbb{R}_+$ so the integral in (78) is the same integrating over \mathbb{R}_- or $\mathbb{R}_- \cup \{-\infty\}$.) Using (15), we also obtain

$$\lim_n \frac{1}{n} \log \frac{1}{\sigma'_1(d_n x)} = \varphi(x),$$

uniformly on each compact subset of \mathbb{R}_+ . Consequently,

$$\lim_{n \in \Lambda} \frac{1}{2n} \log \left(\frac{|Q_{n,2}(x)|}{C_n \sigma'_1(d_n x)} \right)^{1/2} = \frac{1}{4} (\varphi(x) - \mathcal{V}^{\lambda_2^*}(x)),$$

uniformly on each compact subset of \mathbb{R}_+ . From (16) it follows that

$$(79) \quad \lim_{x \rightarrow +\infty} (\varphi(x) - \mathcal{V}^{\lambda_2^*}(x) - 4 \log x) = +\infty.$$

The properties shown above allow us to apply the Lemma and Theorem in [27]². It follows that λ_1^* solves the extremal problem

$$(80) \quad \mathcal{P}^{\lambda_1^*}(x) + \frac{1}{4}(\varphi(x) - \mathcal{V}^{\lambda_2^*}(x)) \begin{cases} = w_1^*, & x \in \text{supp}(\lambda_1^*), \\ \geq w_1^*, & x \in \mathbb{R}_+, \end{cases}$$

for some constant w_1^* , and

$$(81) \quad \lim_{n \in \Lambda} \left(\int \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} C_n \sigma'(d_n x) dx \right)^{1/4n} = e^{-w_1^*}.$$

Moreover, the same results guarantee that $\text{supp}(\lambda_1^*)$ is a compact subset of $[0, +\infty)$ (see comments after [27, Lemma, p.121]). This in turn implies that $\int \log(1 + y^2) d\lambda_1^*(y) < \infty$ and, in particular, $|\lambda_1^*| = 1$ in \mathbb{R}_+ . The continuity of $\mathcal{P}^{\lambda_1^*}$ on $\text{supp}(\lambda_1^*)$ follows from (80), so this potential is also continuous on \mathbb{C} .

Formula (80) may be easily rewritten as

$$(82) \quad 2\mathcal{V}^{2\lambda_1^*}(x) - \mathcal{V}^{\lambda_2^*}(x) + \varphi(x) \begin{cases} = \gamma_1^*, & x \in \text{supp}(2\lambda_1^*), \\ \geq \gamma_1^*, & x \in \mathbb{R}_+, \end{cases}$$

with $\gamma_1^* = 4w_1^* + 2 \int \log(1 + y^2) d\lambda_1^*(y)$, which is the first part of (C) in Lemma 4.1 for the pair $(2\lambda_1^*, \lambda_2^*)$.

The varying discrete measure with respect to which $Q_{n,2}$ is orthogonal, see (12), is

$$\sum_{k=1}^{\infty} \eta_{n,k} \delta_{\xi_{k,n}}(t), \quad \eta_{n,k} = \frac{\beta_k}{|Q_n(\xi_{k,n})|} \int_{\mathbb{R}_+} \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} \frac{C_n \sigma'(d_n x) dx}{x - \xi_{k,n}}.$$

Taking into consideration that $|x - \xi_{k,n}| > |t_1/d_n|$, that $\lim_n d_n^{1/n} = 1$, the compactness of $\text{supp}(\lambda_1^*)$, and (81) one has

$$(83) \quad \lim_{n \in \Lambda} \left(\beta_k \int \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} \frac{C_n \sigma'(d_n x) dx}{x - \xi_{k,n}} \right)^{1/2n} = e^{-2w_1^*}$$

uniformly on k in $\{k : \xi_{k,n} \in K\}$ for every compact $K \subset (-\infty, 0]$. On the other hand, from (76) and the continuity of $\mathcal{P}^{\lambda_1^*}$ it also follows that

$$(84) \quad \lim_{n \in \Lambda} |Q_n(x)|^{1/2n} = e^{-\mathcal{P}^{\lambda_1^*}(x)}$$

Set $g(x) = 2w_1^* - \mathcal{P}^{\lambda_1^*}(x)$. Using (83)–(84) we have

$$(85) \quad \lim_{n \in \Lambda} \eta_{n,k}^{1/n} - e^{-g(\xi_{n,k})} = 0$$

uniformly on k in $\{k : \tau_{n,k} \in K\}$ for every compact $K \subset (-\infty, 0]$.

Relation (85) together with the assumptions of Theorem 2.2 complete the requirements for the use of [35, Theorem 7.1] (see also [36]) except that in our case $\lim_{x \rightarrow -\infty} (g(x) - \log|x|) = 2w_1 \neq +\infty$, while in [35, Theorem 7.1] this limit is required to be $+\infty$. However, the result remains valid with small variations. We cannot assert the $\text{supp}(\lambda_2^*)$ is a compact set (and in

²Actually, in the proof of those results the slightly stronger condition $\lim_{x \rightarrow +\infty} (\varphi(x) - \mathcal{V}^{\lambda_2^*}(x))/4 \log x > 1$ is required; however (79) is also sufficient.

general it is not) and instead of using apriori the potential $\mathcal{P}^{\lambda_2^*}$ to state the conclusion we must make use of $\mathcal{V}^{\lambda_2^*}$. Thus, we deduce that λ_2^* satisfies the equilibrium conditions

$$(86) \quad \mathcal{V}^{\lambda_2^*}(x) - \mathcal{P}^{\lambda_1^*}(x) + 2w_1^* \begin{cases} \leq w_2^*, & x \in \text{supp}(\lambda_2^*), \\ \geq w_2^*, & x \in \text{supp}(\sigma - \lambda_2^*), \end{cases}$$

for some constant w_2^* and

$$(87) \quad \lim_{n \in \Lambda} \left(\int \frac{Q_{n,2}^2(t)}{C_n^2} \frac{1}{|Q_n(t)|} \int Q_n^2(x) \frac{C_n}{|Q_{n,2}(x)|} \frac{\sigma'(d_n x) dx}{|x-t|} d\sigma_{2,n}(t) \right)^{1/2n} = e^{-w_2^*}.$$

(The C_n and C_n^2 , which obviously simplify, come from the need of balancing $Q_{n,2}$ so as to use in the calculations the modified potential $\mathcal{V}^{\nu_{Q_{n,2}}}$ and its limit $\mathcal{V}^{\lambda_2^*}$. We wrote the left hand of (87) in that form to underline this fact.)

Reasoning as in the last part of the proof of Lemma 4.2, the first inequality in (86) on $\text{supp}(\lambda_2^*)$ implies that $\int \log(1+y^2) d\lambda_2^* < +\infty$. Consequently, $|\lambda_2^*| = 1$ on \mathbb{R}_- and $(2\lambda_1^*, \lambda_2^*) \in \mathfrak{M}^*(\sigma)$. Moreover, relations (86) can be rewritten as follows

$$(88) \quad 2\mathcal{V}^{\lambda_2^*}(x) - \mathcal{V}^{2\lambda_1^*}(x) \begin{cases} \leq \gamma_2^*, & x \in \text{supp}(\lambda_2^*), \\ \geq \gamma_2^*, & x \in \text{supp}(\sigma - \lambda_2^*), \end{cases}$$

with $\gamma_2^* = 2w_2^* - 4w_1^* - \int \log(1+y^2) d\lambda_1^*(y)$. Since $(2\lambda_1^*, \lambda_2^*)$ fulfills relations (82) and (88) we have that this pair of measures is the unique extremal solution of Lemma 4.1 (or what is the same in this case of Theorem 2.1). This is true for any subsequence of zero counting measures satisfying (76). So the sequences converge to $\lambda_1/2$ and λ_2 respectively, as stated.

Notice that the limits in (81) and (87) also exist because the constants w_1^*, w_2^* on the right hand are uniquely determined. Using that $\int \log(1+y^2) d\lambda_2 < +\infty$, the definition of C_n , and (14), we obtain

$$(89) \quad \lim_n C_n^{1/n} = e^{\int \log \sqrt{1+y^2} d\lambda_2(y)}.$$

So

$$\lim_n \left(\int \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} \sigma'(d_n x) dx \right)^{1/n} = e^{-(4w_1^* + \int \log \sqrt{1+y^2} d\lambda_2(y))}.$$

From (80) it is easy to check that $w_1 = 4w_1^* + \int \log \sqrt{1+y^2} d\lambda_2(y)$ and (23) follows. Analogously, from (87) and (89) we get

$$\lim_{n \in \Lambda} \left(\int \frac{Q_{n,2}^2(t)}{|Q_n(t)|} \int \frac{Q_n^2(x)}{|Q_{n,2}(x)|} \frac{\sigma'(d_n x) dx}{|x-t|} d\sigma_{2,n}(t) \right)^{1/n} = e^{-2w_2^* + \int \log \sqrt{1+y^2} d\lambda_2(y)}.$$

Using (86), we get $w_2 = 2w_2^* - 4w_1^* - \int \log(1+y^2) d\lambda_2$. Thus,

$$w_1 + w_2 = 2w_2^* - \int \log \sqrt{1+y^2} d\lambda_2(y),$$

as needed to complete the proof of (24) and the theorem. \square

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(Aptekarev) KELDYSH INSTITUTE OF APPLIED MATHEMATICS, MOSCOW, RUSSIA

E-mail address, Aptekarev: **aptekaa@keldysh.ru**

(López-Lagomasino) DEPARTMENT OF MATHEMATICS, UNIVERSIDAD CARLOS III DE MADRID, LEGANÉS,
SPAIN

E-mail address, López: **lago@math.uc3m.es**

(Martínez-Finkelshtein) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALMERÍA, ALMERÍA, SPAIN

E-mail address, Martínez: **andrei@ual.es**